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I apply (i) a classical version of the Ermakov-Lewis procedure and (ii) the strictly isospectral supersymmetric approach to the Schroedinger free fall of the bouncing ball type. In both cases, the Airy function Bi, which in general is eliminated as being unphysical, plays a well-defined role. Relevant plots are displayed.

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I. INTRODUCTION

A Schroedinger quantum particle bouncing on a perfectly reflecting surface in a linear gravitational field is known as the quantum bouncing ball [QBB] [1]. The experimental counterpart is a cold atom dropped onto an "atomic mirror", which can be used for holographic manipulation of atomic beams [2]. Bouncing Bose-Einstein condensates have been observed in the laboratory [3]. In the QBB case, one should solve the Schroedinger equation with the potential

$$\begin{aligned} V_{\rm QBB}(z) &= mgz, \qquad \text{if } z>0 \\ V_{\rm QBB}(z) &= \infty\ , \qquad \text{if } z\leq 0\ . \end{aligned} \tag{1}$$

By the scalings $\mathrm{s} = z/l_g$ and $\mathrm{S} = E/mgl_g$, where $l_g = \left(\frac{\hbar^2}{2m^2g}\right)^{1/3}$ is the "gravitational length" unit, the stationary QBB Schroedinger equation becomes dimensionless

$$\frac{\mathrm{d}^2\psi}{\mathrm{ds}^2} - (\mathrm{s}-\mathrm{S})\psi = 0\ . \tag{2}$$

The general solution is a superposition of Airy functions Ai(s) and Bi(s), but Airy's Bi is discarded for going to infinity at large s. Moreover, the perfectly reflecting boundary requires the wave function be zero at the origin and therefore the physical eigenmodes are written as $\psi_{\rm n}(\mathrm{s}) = \mathrm{N_n Ai(s - S_n)}$, where $\mathrm{N_n}$ is the normalization constant and $\mathrm{S_n}$ are the zeros of the Ai function [1]. In other words, a shift of the Airy's argument is performed placing the Airy zeros at the origin. To the best of the author's knowledge all the previous works in this field made use of only Airy function Ai of shifted argument. The main purpose here is to show that there are two techniques in which the Bi function could still be employed without leading to unphysical results. One of them is the Ermakov-Lewis (EL) procedure, which is presented in section II and the other one is the strictly isospectral supersymmetric (SUSY) approach enclosed in section III. A small conclusion section ends up the work.

II. CLASSICAL EL APPROACH FOR QBB

I will use the version of the EL approach [4] that I introduced in previous works in collaboration [5]. Eq. (2) can be mapped in a known way to the canonical equations for a classical point particle of unit mass, generalized coordinate $\mathrm{q} = \psi$, momentum $\mathrm{p} = \dot{\psi}$, (i.e., velocity $v = \dot{\psi}$), where the dot means total derivative with respects to s, i.e., we identify the coordinate s with the classical Hamiltonian time. Thus, one is led to

$$\dot{\mathrm{q}} \equiv \frac{\mathrm{dq}}{\mathrm{ds}} = \mathrm{p} \tag{3}$$

$$\dot{\mathrm{p}} \equiv \frac{\mathrm{dp}}{\mathrm{ds}} = (\mathrm{s}-\mathrm{S})\mathrm{q}\ . \tag{4}$$

These equations describe the canonical motion for a classical point particle as derived from the time-dependent Hamiltonian of the inverted oscillator type

$$\mathrm{H_{cl}(s)} = \frac{\mathrm{p}^2}{2} - (\mathrm{s}-\mathrm{S})\frac{\mathrm{q}^2}{2}\ . \tag{5}$$

For this classical Hamiltonian the triplet of phase-space functions $T_1 = \frac{\mathrm{p}^2}{2}$, $T_2 = \mathrm{pq}$, and $T_3 = \frac{\mathrm{q}^2}{2}$ forms a dynamical Lie algebra, i.e., $\mathrm{H_{cl}} = \sum_{\mathrm{n}=1}^{3}\mathrm{h_n(s)}T_{\mathrm{n}}(\mathrm{p},\mathrm{q})$, which is closed with respect to the Poisson bracket, namely $\{T_1, T_2\} = -2T_1$, $\{T_2, T_3\} = -2T_3$, $\{T_1, T_3\} = -T_2$. Using this algebra $\mathrm{H_{cl}}$ reads

$$\mathrm{H_{cl}} = T_1 - (\mathrm{s}-\mathrm{S})T_3\ . \tag{6}$$

The Lewis invariant $\mathcal{I}$ belongs to the dynamical algebra, i.e., one can write $\mathcal{I}(\mathrm{s}) = \sum_{\mathrm{r}}\epsilon_{\mathrm{r}}(\mathrm{s})T_{\mathrm{r}}$, and by means of $\frac{\partial\mathcal{I}}{\partial\mathrm{s}} = -\{\mathcal{I},\mathrm{H}\}$ one is led to the following equations for the functions $\epsilon_{\mathrm{r}}(\mathrm{s})$

$$\dot{\epsilon}_{\mathrm{r}} + \sum_{\mathrm{n}}\left[\sum_{\mathrm{m}}\mathrm{C^r_{nm}h_m(s)}\right]\epsilon_{\mathrm{n}} = 0\ , \tag{7}$$

where $\mathrm{C^r_{nm}}$ are the structure constants of the Lie algebra that have been already given above. Thus, we get

$$\begin{aligned} \dot{\epsilon}_1 &= -2\epsilon_2 \\ \dot{\epsilon}_2 &= -(\mathrm{s}-\mathrm{S})\epsilon_1 - \epsilon_3 \\ \dot{\epsilon}_3 &= -2(\mathrm{s}-\mathrm{S})\epsilon_2\ . \end{aligned} \tag{8}$$

The solution of this system can be readily obtained by setting $\epsilon_1 = \rho^2$ giving $\epsilon_2 = -\rho\dot{\rho}$ and $\epsilon_3 = \dot{\rho}^2 + \frac{1}{\rho^2}$, where ρ is the solution of the Milne-Pinney (MP) equation [6], $\ddot{\rho} - (\mathrm{s} - \mathrm{S})\rho = \frac{1}{\rho^3}$. Since Pinney's note in 1950 it is widely known how to write ρ as a function of the two particular solutions of the corresponding parametric oscillator problem. We have followed the method of Eliezer and Gray [7] in order to write $\rho(\mathrm{s})$ as a combination of Airy functions that satisfy the initial conditions as given by those authors. Thus, we used in all our calculations the following formula

$$\rho_1(\mathrm{s}) = \mathrm{N}_1 \left[(\mathrm{Ai}(\mathrm{s}-\mathrm{S}_1) + \mathrm{Bi}(\mathrm{s}-\mathrm{S}_1))^2 + \mathrm{Bi}^2(\mathrm{s}-\mathrm{S}_1)\right]^{1/2}, \tag{9}$$

i.e., we used the two Airy functions corresponding to the ground state. In Eq. (9), $\mathrm{S}_1 = (9\pi/8)^{2/3}$ and $\mathrm{N}_1 = (8\pi^2/9)^{1/6}$ [1]. In terms of the MP solution $\rho(\mathrm{s})$ the Lewis invariant reads

$$\mathcal{I}_{\mathrm{n}}(\mathrm{s}) = \frac{(\rho_{\mathrm{n}}\mathrm{p} - \dot{\rho}_{\mathrm{n}}\mathrm{q})^2}{2} + \frac{\mathrm{q}^2}{2\rho_{\mathrm{n}}^2}$$

$$= \frac{1}{2}\left(\rho_{\mathrm{n}}\dot{\psi}_{\mathrm{n}} - \dot{\rho}_{\mathrm{n}}\psi_{\mathrm{n}}\right)^2 + \frac{1}{2}\left(\frac{\psi_{\mathrm{n}}}{\rho_{\mathrm{n}}}\right)^2 . \tag{10}$$

A plot of $\mathcal{I}_1(\mathrm{s})$ is shown in Fig. 1.

In the EL approach the angular quantities are given by the following formulas [8]

$$\Delta\theta^{\mathrm{d}} = \int_0^T \left[\frac{1}{\rho^2} - \frac{1}{2}\frac{d}{ds'}(\dot{\rho}\rho) + \dot{\rho}^2\right] ds' \tag{11}$$

and

$$\Delta\theta^{\mathrm{g}} = \frac{1}{2}\int_0^T \left[\frac{d}{ds'}(\dot{\rho}\rho) - 2\dot{\rho}^2\right] ds', \tag{12}$$

for the dynamical and geometrical angles, respectively. Thus, the total angle will be

$$\Delta\theta^{\mathrm{t}} = \Delta\theta^{\mathrm{d}} + \Delta\theta^{\mathrm{g}} = \int_0^T \frac{1}{\rho^2} ds' . \tag{13}$$

Plots of all these angles calculated using ρ_1 are displayed in Figs 2,3,4, respectively.

III. STRICTLY ISOSPECTRAL SUSY BOUNCING BALL

Factorizations of one-dimensional Schroedinger operators have been first discussed in the SUSY context by Witten in 1981 [9], and are well known in the mathematical literature in the broader sense of Darboux covariance of Schroedinger equations [10].

In 1984, Mielnik [11] introduced a different factorization of the quantum harmonic oscillator based on the general Riccati solution here denoted by $\mathrm{w_g}$. As a result, Mielnik obtained a one-parameter family of potentials with *exactly* the same spectrum as that of the harmonic oscillator. Mielnik's method offers an interesting possibility to construct families of potentials *strictly* isospectral with respect to the initial (bosonic) one by simply taking into account the most general superpotential (i.e., the general Riccati solution). Thus, in the QBB case one requires $\mathrm{V}_+(\mathrm{s}) = \mathrm{w_g^2} + \frac{\mathrm{dw_g}}{\mathrm{ds}}$, where V_+ is the fermionic partner potential of V_{QBB}. It is easy to see that one particular solution to this equation is $\mathrm{w_p} = \mathrm{w(s)}$, where $\mathrm{w(s)} = -\psi_1'/\psi_1$ is the common Witten superpotential. One is led to consider the following Riccati equation $\mathrm{w_g^2} + \frac{\mathrm{dw_g}}{\mathrm{ds}} = \mathrm{w_p^2} + \frac{\mathrm{dw_p}}{\mathrm{ds}}$, whose general solution can be written down as $\mathrm{w_g(s)} = \mathrm{w_p(s)} + \frac{1}{\mathrm{v(s)}}$, where $\mathrm{v(s)}$ is an unknown function. Using this ansatz, one obtains for the function $\mathrm{v(s)}$ the following Bernoulli equation

$$\frac{\mathrm{dv(s)}}{\mathrm{ds}} - 2\,\mathrm{v(s)}\,\mathrm{w_p(s)} = 1, \tag{14}$$

that has the solution

$$\mathrm{v(s)} = \frac{\mathrm{I_0(s)} + \lambda}{\psi_1^2(\mathrm{s})} . \tag{15}$$

The integral $\mathrm{I_0(s)} = \int_0^{\mathrm{s}} \psi_1^2(\mathrm{y})\,\mathrm{dy}$ is a step-like function as one can see in Fig. 5. On the other hand, $\lambda > 0$ is an integration constant thereby considered as a free parameter, which is a measure of the contribution of the second linearly independent solution, i.e., the Airy Bi in the QBB case, as we argued elsewhere [12]. Thus, $\mathrm{w_g(s)}$ can be written as follows

$$\mathrm{w_g(s;\lambda)} = \mathrm{w_p(s)} + \frac{\mathrm{d}}{\mathrm{ds}}\Big[\ln(\mathrm{I_0(s)} + \lambda)\Big]$$

$$= -\frac{\mathrm{d}}{\mathrm{ds}}\Big[\ln\left(\frac{\psi_1(\mathrm{x})}{\mathrm{I_0(s)} + \lambda}\right)\Big]. \tag{16}$$

Finally, one easily gets the parametric family of potentials

$$\mathrm{V(s;\lambda)} = \mathrm{w_g^2(s;\lambda)} - \frac{\mathrm{dw_g(s;\lambda)}}{\mathrm{ds}}$$

$$= \mathrm{V_{QBB}(s)} - 2\frac{\mathrm{d^2}}{\mathrm{ds^2}}\Big[\ln(\mathrm{I_0(s)} + \lambda)\Big]$$

$$= \mathrm{V_{QBB}(s)} - \frac{4\psi_1(\mathrm{s})\psi_1'(\mathrm{s})}{\mathrm{I_0(s)} + \lambda} + \frac{2\psi_1^4(\mathrm{s})}{(\mathrm{I_0(s)} + \lambda)^2} . \tag{17}$$

All $\mathrm{V(s;\lambda)}$ have the same SUSY partner potential $\mathrm{V_+(s)}$ obtained by deleting the ground state. They may be considered as a sort of intermediates between the bosonic potential $\mathrm{V_{QBB}(s)}$ and the fermionic partner $\mathrm{V_+(s)}$. A plot of $\mathrm{V(s;\lambda)}$ is given in Fig. 6. From Eq. (16) one can infer the ground state wave functions for the potentials $\mathrm{V(s;\lambda)}$ as follows

$$\varphi_1(\mathrm{s};\lambda) = \mathrm{N}(\lambda)\frac{\psi_1(\mathrm{s})}{\mathrm{I}_0(\mathrm{s})+\lambda}\ , \qquad (18)$$

where $\mathrm{N}(\lambda)$ is a normalization factor that can be shown to be of the form $\mathrm{N}(\lambda) = \sqrt{\lambda(\lambda+1)}$. The normalized functions ψ_1 and φ_1 are plotted in Fig. 7.

IV. CONCLUSION

We have shown how Airy's function Bi could find a place in the physics of the quantum bouncing ball through two theoretical procedures connecting the Schroedinger equation with the nonlinear Milne-Pinney equation and Riccati equation, respectively. This may help in gaining further insight in the problem of nonrelativistic quantum free fall.

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[1] J. Gea-Banacloche, Am. J. Phys. **67**, 776 (1999) and references therein; M. Wadati, J. Phys. Soc. Japan **68**, 2543 (1999); R. Onofrio and L. Viola, Phys. Rev. A **53**, 3773 (1996).

[2] M. Morinaga *et al.*, Phys. Rev. Lett. **77**, 802 (1996).

[3] K. Bongs *et al.*, Phys. Rev. Lett. **83**, 3577 (1999).

[4] V. Ermakov, Univ. Izv. Kiev, Series III **9**, 1 (1880); H.R. Lewis, Jr., Phys. Rev. Lett. **18**, 510 (1967); J. Math. Phys. **9**, 1976 (1968).

[5] H. Rosu and J.L. Romero, Nuovo Cimento B **114**, 569 (1999); H. Rosu, P. Espinoza, and M. Reyes, Nuovo Cimento B **114**, 1439 (1999); see also P. Espinoza, e-print math-ph/0002005.

[6] E. Pinney, Proc. Am. Math. Soc. **1**, 681 (1950); W.E. Milne, Phys. Rev. **35**, 863 (1930).

[7] C.J. Eliezer and A. Gray, SIAM (Soc. Ind. Appl. Math.) J. Appl. Math. **30**, 463 (1976).

[8] M. Maamache, Phys. Rev. A **52**, 936 (1995); D.A. Morales, J. Phys. A **21**, L889 (1988); J.M. Cerveró and J.D. Lejarreta, J. Phys. A **22**, L663 (1989).

[9] E. Witten, Nucl. Phys. B **188**, 513 (1981). For review, see F. Cooper, A. Khare and U. Sukhatme, Phys. Rep. **251**, 267 (1995).

[10] V.B. Matveev and M.A. Salle, *Darboux Transformations and Solitons* (Springer, Berlin, 1991).

[11] B. Mielnik, J. Math. Phys. **25**, 3387 (1984).

[12] L.J. Boya *et al.*, Nuovo Cimento B **113**, 409 (1998).

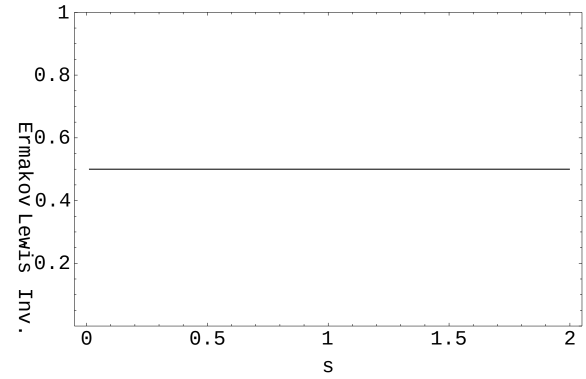

FIG. 1. Ermakov-Lewis invariant $\mathcal{I}_1(\mathrm{s})$ cf. Eq. (10).

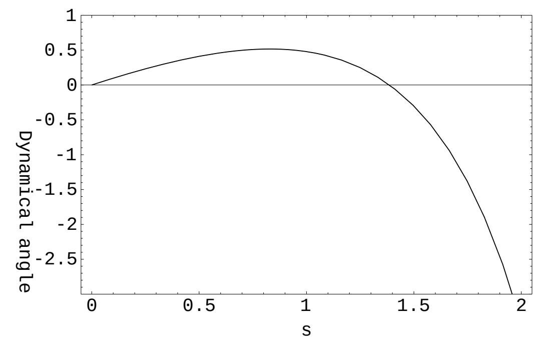

FIG. 2. EL dynamical angle cf. Eq. (11).

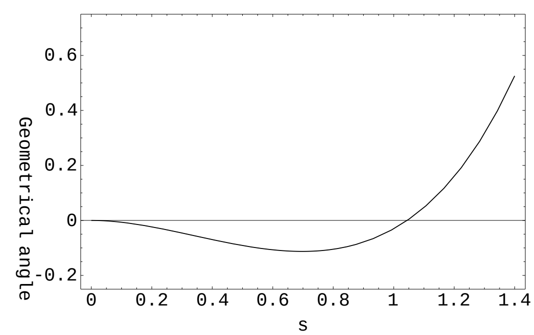

FIG. 3. EL geometric angle cf. Eq. (12).

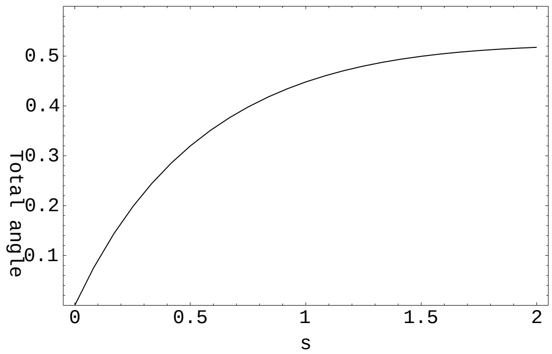

FIG. 4. EL total angle cf. Eq. (13).

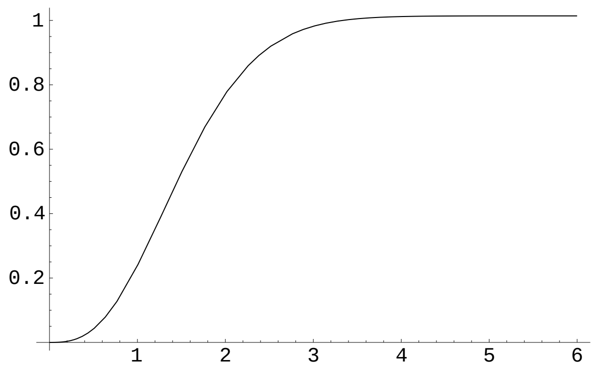

FIG. 5. The integral $I_0(s)$ of the strictly isospectral SUSY QBB.

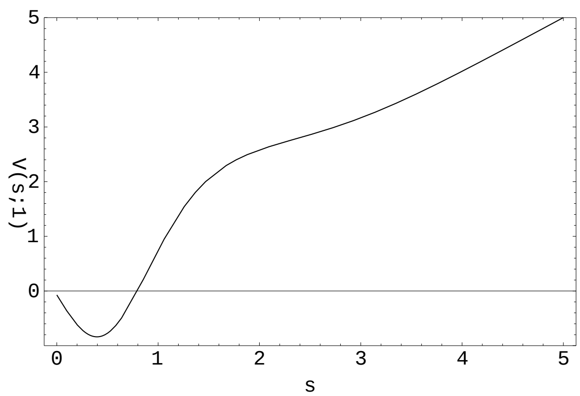

FIG. 6. The strictly isospectral QBB gravitational potential for $\lambda = 1$.

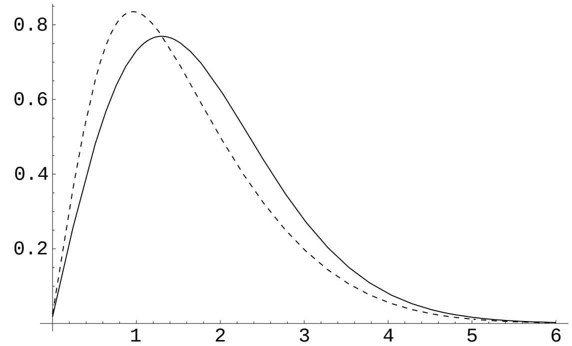

FIG. 7. The normalized wave functions $\psi_1(s)$ (full line) and $\varphi_1(s)$ (dashed line).